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Cubic Spline-Projection Method for Two-Dimensional Integral Equations of Scattering Theory

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This paper investigates a projection method with cubic B-splines for solving two-dimensional Fredholm integral equations of the second kind that arise in scattering theory. Emphasis is placed on the relationship between collocation and Galerkin methods. A mesh grading procedure based on an equidistribution of the nodal points with respect to a measure that combines both the arc length and curvatures is investigated. A test of the numerical procedures is provided by solving the Faddeev integral equation for a model three-boson problem at both bound-state and zero scattering energies. © 1994 Academic Press, Inc.

1. INTRODUCTION

A well-known multidimensional integral equation of scattering theory is found in the work of Faddeev [1] on the quantum three-body problem. Other integral equation formulations of the three-body problem have been developed, for example, by Alt, Grassberger and Sandhas [2]. Over the years considerable effort has been devoted to the accurate numerical solution of these equations, and among the many numerical techniques are those based on product integration [3], Padé approximation [4], variational calculations [5], and projection methods based on cubic spline approximation. Cubic splines have been used to solve one-dimensional integral equations that arise in scattering theory [6, 7], including Faddeev-type integral equations [8–10], as well as partial integrodifferential equations of the Faddeev type [11–13].

The aim of this paper is to investigate projection methods with cubic B-spline approximation to solve two-dimensional integral equations that arise in scattering theory and, in particular, the Faddeev integral equations. Two well-know examples of projection methods are the Galerkin and collocation methods. The Galerkin method requires the evaluation of higher dimensional integrals. For this reason many of the applications of projection methods for solving integral equations of scattering theory make use of the collocation method rather than the Galerkin method. Simpler approaches to the ones described above, such as those based on the Nyström method, may also provide a practical means of solving the integral equations of scattering theory. Indeed, it is possible to construct a Gauss quadrature that is exact for cubic splines and use this quadrature in the Nyström method [14]. An advantage of the Galerkin and collocation methods, however, is that it should be possible to obtain accurate numerical solutions with a smaller number of basis functions. This feature is particularly relevant when solving multidimensional integral equations.

Section 2 gives a mathematical formulation of the cubic spline-projection method for solving two-dimensional integral equations. Section 3 describes an adaptive mesh grading technique. Since the cubic splines can be constructed on nonuniformly spaced points, mesh grading may be viewed as an integral part of the cubic spline-projection method. A brief description of the Faddeev integral equations is given in Section 3. Here it is not the intention to present new results on the three-body problem, but rather to use the Faddeev integral equation as an archetypical equation from scattering theory on which to test the numerical method. The numerical results are given in Section 5 and conclusions in Section 6.

2. APPROXIMATION METHOD

This section describes the numerical procedure for solving two-dimensional integral equations of the second kind,

$$(\mathscr{I} + \mathscr{K}) f = g, \tag{2.1}$$

where $f, g \in C[a, b] \times C[c, d]$ and $\mathscr{I}, \mathscr{K} : [a, b] \times [c, d] \to [a, b] \times [c, d]$. Here $\mathscr{I}$ is the identity operator and $\mathscr{K}$ is the integral kernel defined by

$$\mathscr{K} f(s, t) = \int_a^b \int_c^d K(s, t, s', t')\, f(s', t')\, dt'\, ds'. \tag{2.2}$$

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In what follows it will be assumed that the semi-infinite integrals that arise in scattering theory can be mapped onto a finite interval so that a, b, c, and d are finite. A particular mapping will be discussed in the following section.

The cubic B-splines on $s \in [a, b]$ are constructed as follows [15, 16]: The interval $[a, b]$ is partitioned by m nodal points $\{s_i\}_{i=1}^{m}$, where $a = s_1 < s_2 < \cdots < s_m = b$. Additional points are placed at the ends of this interval, namely, $s_{-2} \leqslant s_{-1} \leqslant s_0 \leqslant s_1$ and $s_m \leqslant s_{m+1} \leqslant s_{m+2} \leqslant s_{m+3}$. Define

$$B_{m,i}^{(1)}(s) = \begin{cases} (s_i - s_{i-1})^{-1}, & s_{i-1} < s \leqslant s_i, \\ 0, & \text{otherwise.} \end{cases} \tag{2.3}$$

The B-splines of order l (degree $l-1$) are generated by the stable iterative method of Cox [16]

$$B_{m,i}^{(l)}(s) = \frac{(s - s_{i-l})\, B_{m,i-1}^{(l-1)}(s) - (s_i - s)\, B_{m,i}^{(l-1)}(s)}{s_i - s_{i-l}}. \tag{2.4}$$

Note that the index on the cubic B-splines runs from $i = 2, \ldots, m+3$. A convenient notation is $B_{m,i}(s) = B_{m,i+2}^{(4)}(s)$ to denote the cubic B-spline which is nonzero over the interval (s_{i-2}, s_{i+2}) so that the index now runs from $i = 0, \ldots, m+1$.

A similar procedure is used to construct cubic B-splines on $t \in [c, d]$. The interval is partitioned by n nodal points such that $c = t_1 < t_2 < \cdots < t_n = d$, and on this partition, together with the extended points $t_{-2} \leqslant t_{-1} \leqslant t_0 \leqslant t_1$ and $t_n \leqslant t_{n+1} \leqslant t_{n+2} \leqslant t_{n+3}$ is constructed the cubic B-splines $\{B_{n,j}(t)\}_{j=0}^{n+1}$.

The next step is to define the approximation operator $\mathscr{S}$, where

$$\mathscr{S}f(s, t) = \sum_{i=0}^{m+1} \sum_{j=0}^{n+1} \lambda_{i,j} B_{m,i}(s)\, B_{n,j}(t). \tag{2.5}$$

The operator $\mathscr{S}$ maps the bivariate function f onto a tensor product of cubic B-splines $\mathscr{S}f$. The bicubic spline approximation has C^2 continuity over the rectangle $[a, b] \times [c, d]$. Since the B-splines have local support it follows that the value of $\mathscr{S}f(s, t)$ depends only on the function $f(s, t)$ in a small neighborhood of (s, t).

In the present context $\mathscr{S}f$ approximates the (unknown) solution f of equation (2.1). Replace f by $\mathscr{S}f$ in Eq. (2.1) and construct a residual function

$$R = \sum_{i=0}^{m+1} \sum_{j=0}^{n+1} \lambda_{i,j} (\mathscr{I} + \mathscr{K})\, B_{m,i} B_{n,j} - g. \tag{2.6}$$

The coefficients $\{\lambda_{i,j}\}$ are now determined from a projection method [17]. Let (Ψ, Θ) denote the usual inner product

$$(\Psi, \Theta) = \int_a^b \int_c^d \Psi(s, t)\, \Theta(s, t)\, dt\, ds. \tag{2.7}$$

Furthermore, let $\{\Psi_\rho\}$ be chosen set of $(m+2) \times (n+2)$ test functions. The coefficients $\{\lambda_{i,j}\}$ are obtained by solving the $(m+2) \times (n+2)$ linear equations

$$(\Psi_\rho, R) = 0, \qquad \rho = 0, \ldots, [(m+2) \times (n+2) - 1]. \tag{2.8}$$

In this paper two examples will be considered, namely,

(i) $\Psi_\rho = B_{m,i} B_{n,j}$, which is the Galerkin method, and

(ii) $\Psi_\rho = \delta(s - u_i)\, \delta(t - v_j)$, where $u_i \in [a, b]$ and $v_j \in [c, d]$, which is the method of collocation.

In what follows it is convenient to define the index notation $\rho = i(n+2) + j$ and $\sigma = k(n+2) + l$. Also the coefficients $\{\lambda_{i,j}\}$ are to be arranged in the form of an extended column vector

$$\lambda = (\lambda_{0,0}, \lambda_{0,1}, \ldots, \lambda_{0,n+1}, \lambda_{1,0}, \ldots, \lambda_{m+1,n+1})^{\mathrm{T}}. \tag{2.9}$$

In the Galerkin approach the column vector λ^{G} is a solution of the matrix equation

$$(\mathbf{M}^{\mathrm{G}} + \mathbf{N}^{\mathrm{G}})\, \lambda^{\mathrm{G}} = \mathbf{e}. \tag{2.10}$$

Here $\mathbf{e}$ is the column vector with elements

$$e_\rho = \int_a^b \int_c^d B_{m,i}(s)\, B_{n,j}(t)\, g(s, t)\, dt\, ds. \tag{2.11}$$

The matrix $\mathbf{N}^{\mathrm{G}}$ has elements defined by four-dimensional moment integrals:

$$N_{\rho,\sigma}^{\mathrm{G}} = \int_a^b \int_c^d \int_a^b \int_c^d B_{m,i}(s)\, B_{n,j}(t)\, K(s, t, s', t') \times B_{m,k}(s')\, B_{n,l}(t')\, dt'\, ds'\, dt\, ds. \tag{2.12}$$

The matrix $\mathbf{M}^{\mathrm{G}}$ can be written

$$M_{\rho,\sigma}^{\mathrm{G}} = P_{i,k} Q_{j,l}, \tag{2.13}$$

where

$$P_{i,k} = \int_a^b B_{m,i}(s)\, B_{m,k}(s)\, ds \tag{2.14}$$

and

$$Q_{j,l} = \int_c^d B_{n,j}(t)\, B_{n,l}(t)\, dt. \tag{2.15}$$

Matrices $\mathbf{P}$ and $\mathbf{Q}$ are banded with elements that can be evaluated explicitly. In the case of uniformly spaced nodal

points, with spacing h, these matrices are completely specified from a 4×4 matrix

$$\frac{1}{80640h}\begin{bmatrix} 20 & 129 & 60 & 1 \\ 129 & 1188 & 933 & 60 \\ 60 & 933 & 1188 & 129 \\ 1 & 60 & 129 & 20 \end{bmatrix}. \tag{2.16}$$

In the collocation approach the column vector λ^C is a solution of the matrix equation

$$(\mathbf{M}^C + \mathbf{N}^C)\,\lambda^C = \mathbf{g}, \tag{2.17}$$

where

$$g_\rho = g(u_i, v_j), \tag{2.18}$$

$$M^C_{\rho,\sigma} = B_{m,k}(u_i)\, B_{n,l}(v_j), \tag{2.19}$$

and matrix $\mathbf{N}^C$ has elements defined by two-dimensional moment integrals

$$N^C_{\rho,\sigma} = \int_a^b \int_c^d K(u_i, v_j, s, t)\, B_{m,k}(s)\, B_{n,l}(t)\, dt\, ds. \tag{2.20}$$

The collocation method has some freedom in the choice of the collocation points $\{u_i\}_{i=0}^{m+1}$ and $\{v_j\}_{j=0}^{n+1}$. In practice the placement of the collocation points is restricted by the Schoenberg–Whithney theorem [18, p. 200, Theorem XIII.1] which requires that $u_i \in (s_{i-2}, s_{i+2})$ and $v_j \in (t_{j-2}, t_{j+2})$. The choice adopted in the present paper is to place one collocation point at each nodal point in the rectangle $[a, b] \times [c, d]$. Additional collocation points are placed at the midpoint of each end interval. Thus

$$\begin{aligned} &u_0 = s_1, \qquad u_1 = \tfrac{1}{2}(s_1 + s_2), \\ &u_i = s_i, \qquad i = 2, \ldots, m-1, \\ &u_m = \tfrac{1}{2}(s_m + s_{m+1}), \qquad u_{m+1} = s_m, \end{aligned} \tag{2.21}$$

and

$$\begin{aligned} &v_0 = t_1, \qquad v_1 = \tfrac{1}{2}(t_1 + t_2), \\ &v_j = t_j, \qquad j = 2, \ldots, n-1, \\ &v_n = \tfrac{1}{2}(t_n + t_{n+1}), \qquad v_{n+1} = t_n. \end{aligned} \tag{2.22}$$

The homogeneous form [19] of Eq. (2.1) is the integral equation

$$\eta \Phi = -\mathscr{K}\Phi, \tag{2.23}$$

where $\Phi \in C[a, b] \times C[c, d]$ is an eigenfunction corresponding to the eigenvalue η. In general there will be an infinite sequence $\{\eta_i\}$ and corresponding $\{\Phi_i\}$ that satisfy Eq. (2.23). Using the approximation

$$\mathscr{S}\Phi(s, t) = \sum_{i=0}^{m+1} \sum_{j=0}^{n+1} \lambda_{i,j} B_{m,i}(s)\, B_{n,j}(t) \tag{2.24}$$

leads to the generalized eigenvalue problem

$$\tilde{\eta}\mathbf{M}\lambda = -\mathbf{N}\lambda, \tag{2.25}$$

where matrices $\mathbf{M}$ and $\mathbf{N}$ correspond to either the Galerkin or collocation methods. The $(m+2) \times (n+2)$ eigenvalues $\{\tilde{\eta}_i\}$ may be considered an approximation of the sequence $\{\eta_i\}$, and $\{\mathscr{S}\Phi_i\}$ the corresponding eigenfunctions.

The integrals in Eqs. (2.11), (2.12), and (2.20) are evaluated using a standard Gauss–Legendre quadrature formula [20].

3. ADAPTIVE MESH GRADING TECHNIQUE

This section describes a technique for choosing the position of the $m \times n$ points on the rectangle $[a, b] \times [c, d]$. A simple choice, which is appropriate if the solution of the integral equation is a slowly varying function of the independent variables s and t, is to space the grid points uniformly throughout the entire rectangular region. On the other hand, if the solution is not slowly varying in some region of the grid then it may be possible to improve the accuracy of the numerical solution by concentrating some of the grid points in this region. A procedure that redistributes a given number of grid points is usually referred to as an adaptive mesh grading (AMG). The approach used in this paper is to equi-distribute the nodal points with respect to a measure that combines both the arc length and the curvatures of the solution function. This approach is taken from the AMG procedure for solving one-dimensional integral equations investigated by Eyre and Wright [21]. A description of this procedure will now be given.

Let $z(s) \in C^2[a, b]$ be the cubic B-spline approximation defined on the partition $a = s_1 < s_2 < \cdots < s_m = b$. Let

$$a_i = \int_{s_{i-1}}^{s_i} [1 + (z')^2]^{1/2}\, ds, \tag{3.1}$$

$$c_i = \int_{s_{i-1}}^{s_i} |z''|\, [1 + (z')^2]^{-3/2}\, ds \tag{3.2}$$

be the arc length and curvature of z over the subinterval $[s_{i-1}, s_i]$. The coefficients $\{a_i\}$ and $\{c_i\}$ are normalised so that

$$\sum_{i=2}^{m} a_i = \sum_{i=2}^{m} c_i = 1. \tag{3.2}$$

Now define

$$k_i = (1-\omega)\, a_i + \omega c_i, \tag{3.3}$$

where ω is a weighting factor. The idea behind the AMG technique is to redistribute the points $\{s_i\}$ with respect to this weighted combination of the arc length and curvature.

In order to damp extreme values a weighted means quantity is introduced

$$\begin{aligned} \beta_2 &= \tfrac{1}{3}(2k_2 + k_3), \\ \beta_i &= \tfrac{1}{4}(k_{i-1} + 2k_i + k_{i+1}), \qquad i = 3, \ldots, m-1, \\ \beta_m &= \tfrac{1}{3}(k_{m-1} + 2k_m). \end{aligned} \tag{3.4}$$

The measure of arc length and curvatures is obtained by integrating β,

$$\Gamma_i = \sum_{j=2}^{i} \beta_j, \tag{3.5}$$

with $\Gamma_1 = 0$. The values obtained from $\Gamma_i = \Gamma(s_i)$ are now used in inverse form, $s_i = s(\Gamma_i)$. Let

$$\Gamma_i^+ = \frac{i}{(m-1)}\, \Gamma_m. \tag{3.6}$$

The new nodal points

$$\hat{s}_i = s(\Gamma_i^+) \tag{3.7}$$

correspond to an equidistribution of the quantities β_i over the $m-1$ subintervals.

The above procedure is repeated until the new nodal points $\{\hat{s}_i\}$ have converged. Convergence is obtained when

$$\sigma = \frac{1}{(m-1)} \left[\sum_{i=2}^{m} (\beta_i - \bar{\beta})^2 \right]^{1/2}, \tag{3.8}$$

is less than some specified tolerance. Here $\bar{\beta}$ is the average value of $\{\beta_i\}$.

Once a converged set of new points has been constructed the integral equation is again solved and the procedure repeated until the solution function does not change significantly.

4. FADDEEV INTEGRAL EQUATION

This section gives a brief description of the Faddeev integral equations. The notation of Kloet and Tjon [4] will be followed. Let p and q be magnitudes of the Jacobi momentum variables. For identical particles the S-wave Faddeev integral equation becomes

$$\begin{aligned} U(p, q) = {} & \frac{32\pi}{\sqrt{3}\, qq_i} \int_{A(q, q_i)}^{B(q, q_i)} p'\, dp' t \\ & \times (p, \sqrt{p'^2 + q_i^2 - q^2}; s - q^2)\, \psi(p') \\ & - \frac{16\pi}{\sqrt{3}\, q} \int_0^\infty q'\, dq' \int_{A(q, q')}^{B(q, q')} p'\, dp' t \\ & \times (p, \sqrt{p'^2 + q'^2 - q^2}; s - q^2) \frac{U(p', q')}{p'^2 + q'^2 - E}, \end{aligned} \tag{4.1}$$

where $A(q, q') = |2q - q'|/\sqrt{3}$, $B(q, q') = |2q + q'|/\sqrt{3}$, and E is the complex three-body energy. The two-particle t-matrix satisfies a Lippmann–Schwinger integral equation,

$$\begin{aligned} t(p, p'; \varepsilon) = {} & v(p, p') - 4\pi \int_0^\infty \frac{p''^2\, dp''}{p''^2 - \varepsilon} \\ & \times v(p, p'')\, t(p'', p'; \varepsilon), \end{aligned} \tag{4.2}$$

where ε is the complex two-body energy. It will be assumed that the potential $v(p, p')$ can support a single two-body bound state with binding energy b, where $\mathrm{Re}(b) > 0$. In this case the bound state wavefunction ψ is given by

$$\psi(p) = \frac{\phi(p)}{(p^2 + b)}, \tag{4.3}$$

where ϕ satisfies the homogeneous integral equation

$$\phi(p) = -4\pi \int_0^\infty \frac{p'^2\, dp'}{p'^2 + b}\, v(p, p')\, \phi(p'). \tag{4.4}$$

The function t contains a bound state pole at $\varepsilon = -b$. Thus

$$t(p, p'; \varepsilon) = \frac{\tilde{t}(p, p'; \varepsilon)}{b + \varepsilon}, \tag{4.5}$$

where $\tilde{t}$ is a function that does not contain this pole. In order to compute the solution of Eq. (4.1) it is necessary to take this singularity into account. Define

$$\Omega(p, q) = q^2 U(p, q). \tag{4.6}$$

At scattering threshold the three-body energy is $E = -b$. The integral in the inhomogeneous term of Eq. (4.1) can be evaluated so that Eq. (4.6) becomes

$$\Omega(p,q)=\frac{128\pi}{3\sqrt{3}}\tilde{t}\left(p,\frac{q}{\sqrt{3}};-b-q^2\right)\psi\left(\frac{2}{\sqrt{3}}q\right)$$
$$-\frac{16\pi}{\sqrt{3}\,q}\int_0^\infty\frac{dq'}{q'}\int_{A(q,q')}^{B(q,q')}p'\,dp'\tilde{t}$$
$$\times\left(p,\sqrt{p'^2+q'^2-q^2};-b-q^2\right)\frac{\Omega(p',q')}{p'^2+q'^2+b}. \tag{4.7}$$

The scattering length, α, is given by

$$\alpha=-\frac{\pi}{8}\frac{\Omega(0,0)}{\phi(0)}. \tag{4.8}$$

Bound states of the three-body system are described by eigenfunctions Φ of the homogeneous integral equation

$$\eta\Phi(p,q)=-\frac{16\pi}{\sqrt{3}\,q}\int_0^\infty q'\,dq'\int_{A(q,q')}^{B(q,q')}p'\,dp't$$
$$\times\left(p,\sqrt{p'^2+q'^2-q^2};s-q^2\right)\frac{\Phi(p',q')}{p'^2+q'^2-E}, \tag{4.9}$$

corresponding to the eigenvalue $\eta=1$. Only the eigenvalue $\eta=1$ corresponds to a solution of the Schrödinger equation. This eigenvalue occurs at discrete values of $E=-B_i$, where B_i is a (positive) three-body binding energy and i labels the state.

5. NUMERICAL RESULTS

In order to apply the spline–Galerkin method it is convenient to map the momentum variables p and q onto a finite interval. Let dimensionless variables s and t be defined by the mapping

$$p=\zeta\left(\frac{1+s}{1-s}\right),\qquad q=\nu\left(\frac{1+t}{1-t}\right), \tag{5.1}$$

where ζ and ν are constant parameters. Both $s,\,t\in[-1,1]$ and at the midpoint of the interval $p=\zeta$ and $q=\nu$. The parameters ζ and ν are chosen so that the solution of the integral equation is distributed in a reasonable way over the square $[-1,1]\times[-1,1]$. Since this mapping will also redistribute the nodal points in the p and q variables it may be viewed as a rough type of mesh grading. A more refined mesh grading procedure was described in Section 3. In the problems considered below a reasonable choice for these parameters is $\zeta=\nu=1\ \text{fm}^{-1}$.

The two-body interaction is chosen to be a non-separable potential given by the sum of Yukawa functions

$$v(r)=V_R\frac{e^{-\mu_R r}}{r}-V_A\frac{e^{-\mu_A r}}{r}, \tag{5.2}$$

TABLE I

Potential Parameters

Potential	V_R(MeV·fm)	V_A(MeV·fm)	μ_R(fm^{-1})	μ_A(fm^{-1})
Delves	0	49.7616×1.58	0	$(1.58)^{-1}$
MT-III	1438.720	626.885	3.11	1.55
MT-V	1438.4812	570.3316	3.11	1.55

where V_R and V_A are positive. Three cases are considered: the attractive Delves [22] potential, and the two-term Malfliet–Tijon [23] (MT-II and MT-V) potentials. The potential parameters are chosen to agree with Ref. [11] and are tabulated in Table I for convenience. (In all calculations $\hbar^2/m=41.468\ \text{MeV}\cdot\text{fm}^2$.) The two-term potentials have a strong repulsion and therefore more structure than the single term attractive Delves potential.

The discussion of the two-body subsystem will be restricted to a single partial wave with angular momentum $l=0$. In this case the Fourier transform of the local potential $v(r)$ is

$$v(p,p')=\frac{1}{2\pi^2}\int_0^\infty j_0(pr)\,j_0(p'r)\,v(r)\,r^2\,dr, \tag{5.3}$$

where j_0 is the spherical Bessel function of the first kind of order 0. The resulting Lippmann–Schwinger integral equation (4.2) is solved numerically using a degenerate kernel method [24]. For this purpose a sufficiently fine mesh is used so that the error in the two-body t-matrix does not contribute significantly to the error in the three-body calculations.

Tables II and III show numerical results for the three-

TABLE II

Three Boson Binding Energies in Mega Electron Volts for the Delves Potential

	B_1		B_2	
$(n+2)^2$	(C)	(G)	(C)	(G)
36	58.014	50.381	—	8.905
49	50.867	50.413	—	8.760
64	50.715	50.494	7.905	8.515
81	50.610	50.510	8.270	8.630
100	50.564	50.512	8.574	8.650
121	50.542	50.512	8.664	8.650
144	50.532		8.680	
169	50.527		8.681	
196	50.521		8.672	
225	50.519		8.669	
256	50.519		8.665	
289	50.517		8.662	
324	50.516		8.660	
Friar *et al.* [11]		50.5093		8.652

TABLE III

Three Boson Binding Energy in Mega Electron Volts for the MT-V Potential

$(n+2)^2$	(C)	(G)
36	7.523	7.597
49	6.926	7.563
64	7.655	7.541
81	7.641	7.526
100	7.590	7.548
121	7.565	7.540
144	7.554	
169	7.550	
196	7.546	
225	7.543	
256	7.542	
289	7.541	
324	7.540	
Friar *et al.* [11]		7.540

body binding energies. In a three-boson system the Delves potential supports two bound states. The most accurately known values have been obtained by Friar *et al.* [11]; the ground state energy is $B_1 = 50.5093$ MeV and the first excited state energy is $B_2 = 8.652$ MeV. Numerical results obtained using both the collocation (C) and Galerkin (G) methods show a monotonic convergence towards these values. For a given mesh size the most accurate results are obtained using the Galerkin method. The value in Ref. [11] for the MT-V potential is $B = 7.540$ MeV. In this case the convergence shown in Table III does not appear to be monotonic. Nevertheless it can be seen that the most accurate results are again obtained using the Galerkin method.

Table IV shows numerical results for the three-boson scattering length α. For the MT-III potential Payne *et al.* [12] give $\alpha = 26.03\ \text{fm}^{-1}$. Both the collocation (C) and Galerkin (G) methods indicate numerical convergence towards this value, i.e., the error decreases as n increases to within about 8% of the reference solution.

TABLE IV

Scattering Length α per Femtometer for the MT-III Potential

$(n+2)^2$	(C)	(G)
36	100.6	61.3
49	58.3	41.2
64	35.9	31.1
81	33.2	28.7
100	31.3	24.8
121	30.3	
144	29.7	
169	29.2	
196	28.8	
225	28.5	
256	28.2	
Payne *et al.* [12]		26.03

As a further check on the algorithm the scattering length has also been calculated for the MT-V potential in Table I. For $n = 16$ the collocation method gives a value of $\alpha = 35.1\ \text{fm}^{-1}$ which compares reasonably well with the result given by Payn *et al.* of $\alpha = 34.9 \pm 0.2\ \text{fm}^{-1}$.

Consider the solution of Eq. (4.7) over the square $[-1, 1] \times [-1, 1]$. Along the edges $s = 1$ and $t = 1$ the solution $\Omega \to 0$. The nodal points may therefore be expected to be equidistributed with respect to the weighted combination of arc length and curvatures of the approximate bicubic spline solution Ω_n. Along the edges $s = -1$ and $t = -1$, however, the situation is different because the approximate solution Ω_n is expected to exhibit some structure. Along these edges an AMG procedure described in Section 3 can be used to better distribute the nodal points. Because the rectangular grid is uniquely defined by the set of points $\{s_i, t_j\}$ it is sufficient to construct the nodal points $\{s_i\}$ and $\{t_j\}$. The new nodal points are constructed as follows: An AMG procedure is applied to the spline functions $z_1(s) = \Omega_n(p(s), 0)$ and $z_2(t) = \Omega_n(0, q(t))$. The points $\{\hat{s}_i\}$ and $\{\hat{t}_j\}$ obtained from this procedure are now the new grid points $(\hat{s}_i, \hat{t}_j)$. Also σ_1 and σ_2 are defined by Eq. (3.8) for the spline functions z_1 and z_2, respectively. As pointed out in Ref. [21] a blind application of the AMG technique is not advisable because not all tolerance values for σ lead to a converged solution. A careful choice of the tolerance values is therefore required. The nodal points that lie outside the square $[-1, 1] \times [-1, 1]$ do not affect the bicubic spline interpolation and are left unchanged by the AMG procedure.

In order to test the accuracy of numerical solution on a coarse grid the result obtained from the collocation method using a fine mesh with $n = 12$ was used as a reference solution. An estimate of the global error in the numerical solution $\Omega_n(p(s), q(t))$ is obtained from the L^2-norm of the error function

$$\Delta\Omega = \Omega_n - \Omega_{\text{ref}}, \tag{5.4}$$

TABLE V

Collocation Solution with $(n+2)^2 = 49$ Using $\omega = 0$ and I Iterates of the AMG

I	α	$\|\Delta\Omega\|_2$	σ_1	σ_2
0	58.3	23.4	4.6(−2)	4.9(−2)
1	23.0	15.9	4.7(−2)	4.8(−2)
2	34.4	17.0	4.3(−2)	3.7(−2)
3	26.9	17.3	4.8(−2)	4.8(−2)
4	35.6	17.1	4.4(−2)	4.8(−2)
5	27.0	17.0	4.7(−2)	4.8(−2)

TABLE VI

Collocation Solution with $(n+2)^2 = 49$ Using $\omega = 0.25$ and I Iterates of the AMG

I	α	$\|\Delta\Omega\|_2$	σ_1	σ_2
0	58.3	23.4	4.2(−2)	4.2(−2)
1	36.8	16.7	4.3(−2)	4.1(−2)
2	41.4	17.0	4.2(−2)	4.2(−2)
3	37.2	16.7	4.3(−2)	3.9(−2)
4	39.0	17.0	3.8(−2)	3.8(−2)
5	37.8	17.2	4.0(−2)	3.7(−2)

TABLE VIII

Collocation Solution with $(n+2)^2 = 64$ Using $\omega = 0.25$ and I Iterates of the AMG

I	α	$\|\Delta\Omega\|_2$	σ_1	σ_2
0	35.9	16.3	3.0(−2)	3.0(−2)
1	29.0	12.0	2.9(−2)	3.0(−2)
2	30.6	11.8	2.9(−2)	2.9(−2)
3	30.2	11.7	2.8(−2)	2.9(−2)
4	30.5	10.6	2.8(−2)	3.0(−2)
5	30.7	10.5	2.7(−2)	2.9(−2)

where Ω_{ref} is the reference solution. The L^2-norm is defined by

$$\|\Delta\Omega\|_2 = \int_{-1}^{1}\int_{-1}^{1} |\Delta\Omega(p(s), q(t))|^2 \, ds \, dt. \qquad (5.5)$$

Tables V and VI show results for the AMG procedure using the collocation method with $(n+2)^2 = 49$, while Tables VII and VIII show results for $(n+2)^2 = 64$. The number of iterations is denoted by I, with $I = 0$ being the preliminary solution. Results using both arc length ($\omega = 0$) and a combination of arc length and curvatures ($\omega = 0.25$) are shown.

Convergence of the iterative procedure is defined using tolerance values for σ_1 and σ_2. We remark that the computed values of σ_1 and σ_2 do not depend on any reference solution but reflect a change in the distribution of the points with respect to the measure as given by Eq. (3.8). Not all choices of ω lead to a converged solution. For example, in Table V we see that for $\omega = 0$ the values of σ_1 and σ_2 do not decrease significantly in the first few iterates. On the other hand, the results of Table VI show that for $\omega = 0.25$ a tolerance value of $\sigma = 0.04$ would lead to convergence after $I = 5$ iterates of the AMG.

The AMG procedure leads to an improved approximation of the scattering solution as indicated by the computed scattering length α and the L^2-norm of the error function $\|\Delta\Omega\|_2$.

TABLE VII

Collocation Solution with $(n+2)^2 = 64$ Using $\omega = 0$ and I Iterates of the AMG

I	α	$\|\Delta\Omega\|_2$	σ_1	σ_2
0	35.9	16.3	3.0(−2)	3.0(−2)
1	12.1	11.4	3.0(−2)	3.0(−2)
2	9.4	10.5	2.9(−2)	2.9(−2)
3	11.6	10.5	2.8(−2)	2.9(−2)
4	12.1	10.5	2.8(−2)	2.9(−2)
5	12.7	10.5	2.8(−2)	2.9(−2)

6. CONCLUSIONS

The projection method using cubic B-splines has been used to solve the two-dimensional Faddeev integral equation. This approach reduces the problem to one of evaluating multidimensional integrals and solving a system of algebraic equations. The number of algebraic equations increases with the number of grid points. Results have been obtained using both Galerkin and collocation methods. On a coarse grid the Galerkin technique gives the most accurate numerical solutions, but this advantage is offset by the fact that the method requires the evaluation of higher dimensional integrals. In practice these integrals are evaluated by numerical quadrature.

An alternative approach is to improve the accuracy of the collocation method by using an AMG technique. Since the solution function is approximated as a bicubic spline, the nodal points can be equi-distributed with respect to a measure that combines both arc length and curvatures of the approximate solution to the integral equation. The AMG technique has been applied iteratively to obtain a converged solution. The iterative procedure does not necessarily increase the computer time because the truncation error may be improved so that accurate solutions can be obtained with a smaller number of grid points. Indeed, for the example considered in this paper it is possible to improve the accuracy of the approximate solution on a coarse grid.

We remark that the AMG technique has not been applied to the Galerkin method because this method requires the evaluation of higher dimensional integrals, and, unlike the collocation method, these integrals are costly to evaluate.

The cubic-spline projection method has been shown to be both an accurate and an efficient method for solving the two-dimensional Faddeev integral equation at bound state and zero scattering energies. For scattering energies above the three-body breakup threshold the kernel of the Faddeev integral equation contains logarithmic singularities [9]. An important feature of the present method is that the multidimensional integrals can be evaluated accurately

using standard numerical quadratures. Thus logarithmic singularities in the kernel do not pose any major difficulty. In the case of realistic forces involving many partial waves it is known that the solution of the Faddeev equation does result in large systems of algebraic equations. An approach that seeks to reduce the number of algebraic equations, such as the method described in this paper, should be of value in the numerical treatment of such problems. Finally, the cubic spline-projection method can be extended to treat integral equations in more than two dimensions. Integral equations of this type arise, for example, in many-body scattering theory.

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